

Exact Solution of Noncommutative $U(1)$ Gauge Theory in 4-Dimensions

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Abstract

Noncommutative $U(1)$ gauge theory on the Moyal-Weyl space $\mathbf{R}^2 \times \mathbf{R}_\theta^2$ is regularized by approximating the noncommutative spatial slice \mathbf{R}_θ^2 by a fuzzy sphere of matrix size L and radius R . Classically we observe that the field theory on the fuzzy space $\mathbf{R}^2 \times \mathbf{S}_L^2$ reduces to the field theory on the Moyal-Weyl plane $\mathbf{R}^2 \times \mathbf{R}_\theta^2$ in the flattening continuum planar limits $R, L \rightarrow \infty$ where the ratio $\theta^2 = R^2/|L|^{2q}$ is kept fixed with $q > \frac{3}{2}$. The effective noncommutativity parameter is found to be given by $\theta_{eff}^2 \sim 2\theta^2(\frac{L}{2})^{2q-1}$ and thus it corresponds to a strongly noncommuting space. In the quantum theory it turns out that this prescription is also equivalent to a dimensional reduction of the model where the noncommutative $U(1)$ gauge theory in 4 dimensions is shown to be equivalent in the large L limit to an ordinary $O(M)$ non-linear sigma model in 2 dimensions where $M \sim 3L^2$. The Moyal-Weyl model defined this way is also seen to be an ordinary renormalizable theory which can be solved exactly using the method of steepest descents. More precisely we find for a fixed renormalization scale μ and a fixed renormalized coupling constant g_r^2 an $O(M)$ -symmetric mass, for the different components of the sigma field, which is non-zero for all values of g_r^2 and hence the $O(M)$ symmetry is never broken in this solution. We obtain also an exact representation of the beta function of the theory which agrees with the known one-loop perturbative result.

1 Introduction

We propose in this article to reconsider the problem of quantum $U(1)$ gauge theory in 4-dimensions where spacetime is noncommutative. In particular we will consider the simple case where only two spatial directions are noncommutative and thus avoiding potential problems with unitarity and causality. Towards the end of regularizing this model we replace the noncommutative Moyal plane with a fuzzy sphere, i.e with a $(L+1) \times (L+1)$ matrix model where $a = \frac{1}{L+1}$ is essentially a lattice-spacing-like parameter. The fuzzy sphere \mathbf{S}_L^2 has two cut-offs, a UV cut-off L (the matrix size) and an IR cut-off R (the radius) which both preserve Lorentz, gauge and chiral symmetries, and which allows us to view the noncommutative Moyal plane as a sequence of matrix models $Mat_{L+1}(R), \dots, Mat_{L'+1}(R'), \dots, Mat_{L''+1}(R'')$ with the two parameters L and R ever increasing ($L \leq L' \leq L''$, $R \leq R' \leq R''$) while (for example) the ratio $R/L = \theta/2$ is kept fixed. In this way one can immediately see that Lorentz symmetry is only lost at the strict limit in the sense that the original $SO(3)$ symmetry is reduced to an $SO(2)$ symmetry while the noncommutativity parameter θ^2 in this prescription is equal to the volume of spacetime per point (in here this is given by the area of the sphere divided by the number of points, i.e $\pi\theta^2 = \frac{4\pi R^2}{L^2}$).

In this section we will first recall few results from noncommutative perturbative gauge theory which will be useful to us in what will follow in this article [1, 2]. The basic noncommutative gauge theory actions of interest to us in this article are matrix models of the form [1]

$$S_\theta = \frac{\theta^d}{4g^2} Tr \hat{F}_{ij}^2 = \frac{\theta^d}{4g^2} Tr \sum_{i,j} \left(i[\hat{D}_i, \hat{D}_j] - \frac{1}{\theta^2} (B^{-1})_{ij} \right)^2. \quad (1)$$

$i, j = 1, \dots, d$, B^{-1} is assumed in here to be an invertible tensor (which in 2 dimensions is $(B^{-1})_{ij} = (\epsilon^{-1})_{ij} = -\epsilon_{ij}$), and θ has dimension of length so that the operators \hat{D}_i 's have dimension of $(\text{length})^{-1}$. The coupling constant g is of dimension $(\text{mass})^{2-\frac{d}{2}}$. The trace is taken over some infinite dimensional Hilbert space \mathbf{H} and hence $Tr[\hat{D}_i, \hat{D}_j]$ is $\neq 0$ in general. In general Tr is equal to the trace over coherent states (corresponding to

spacetime) times the trace over the gauge group if any (in here this is simply $U(1)$) . The sector of this matrix theory which corresponds to a noncommutative gauge field on \mathbf{R}_θ^d is defined by the configurations [1]

$$\hat{D}_i = -\frac{1}{\theta^2}(B^{-1})_{ij}\hat{x}_j + \hat{A}_i, \quad \hat{A}_i^\dagger = \hat{A}_i, \quad (2)$$

where the components \hat{x}_i 's can be identified with those of a background noncommutative gauge field whereas \hat{A}_i 's are identified with the components of the dynamical $U(1)$ noncommutative gauge field . \hat{x}_i 's can also be interpreted as the coordinates on the noncommutative space \mathbf{R}_θ^d satisfying the usual commutation relation

$$[\hat{x}_i, \hat{x}_j] = i\theta^2 B_{ij}. \quad (3)$$

Derivations on this \mathbf{R}_θ^d will be taken for simplicity to be defined by

$$\hat{\partial}_i = -\frac{i}{\theta^2}(B^{-1})_{ij}\hat{x}_j = -\hat{\partial}_i^\dagger, \quad [\hat{\partial}_i, \hat{x}_j] = \delta_{ij}, \quad [\hat{\partial}_i, \hat{\partial}_j] = \frac{i}{\theta^2}(B^{-1})_{ij}. \quad (4)$$

$U(1)$ gauge transformations which leave the action (1) invariant are implemented by unitary matrices $U = \exp(i\Lambda)$, $UU^\dagger = U^\dagger U = 1$, $\Lambda^\dagger = \Lambda$ which act on the Hilbert space \mathbf{H} as follows . The covariant derivative $\hat{D}_i = -i\hat{\partial}_i + \hat{A}_i^\dagger T_a$ and curvature $\hat{F}_{ij} = i[\hat{D}_i, \hat{D}_j] - \frac{1}{\theta^2}(B^{-1})_{ij} = [\hat{\partial}_i, \hat{A}_j] - [\hat{\partial}_j, \hat{A}_i] + i[\hat{A}_i, \hat{A}_j]$ transform as $\hat{D}_i \rightarrow U \hat{D}_i U^\dagger$ (i.e $\hat{A}_i \rightarrow U \hat{A}_i U^\dagger - iU[\hat{\partial}_i, U^\dagger]$), $\hat{F}_{ij} \rightarrow U \hat{F}_{ij} U^\dagger$. By virtue of (2), (3) and (4) it is not difficult to show that the matrix action (1) is precisely the usual noncommutative gauge action on \mathbf{R}_θ^d with a star product defined by the parameter $\theta^2 B^{ij}$, i.e

$$S_\theta = \frac{1}{4g^2} \int d^d x F_{ij}^2; \quad F_{ij} = \partial_i A_j - \partial_j A_i + i\{A_i, A_j\}_*. \quad (5)$$

Quantization of the matrix models (1) consists usually in quantizing the models (5) . This generally makes good sense at one-loop but not necessarily at higher loops which we still do not know how to study systematically . Let us concentrate in the rest of this introduction on the $U(1)$ model in $d = 4$. The one-loop effective action can be easily obtained (for example) in the Feynman-'t Hooft background field gauge where $A_i = A_i^{(0)} + A_i^{(1)}$ one finds the result [3]

$$\Gamma_\theta = S_\theta[A^{(0)}] - \frac{1}{2} \text{Tr}_d \text{TR} \text{Log} \left((\mathcal{D}^{(0)})^2 \delta_{ij} + 2i\mathcal{F}_{ij}^{(0)} \right) + \text{TR} \text{Log}(\mathcal{D}^{(0)})^2, \quad (6)$$

where the operators $(\mathcal{D}^{(0)})^2 = \mathcal{D}_i^{(0)} \mathcal{D}_i^{(0)}$, $\mathcal{D}_i^{(0)}$ and $\mathcal{F}_{ij}^{(0)}$ are defined through a star-commutator and hence even in the $U(1)$ case (which is of most interest in here anyway) the action of these operators is not trivial, viz for example $\mathcal{D}_i^{(0)}(A_j^{(1)}) \equiv [D_i^{(0)}, A_j^{(1)}]_* = -i\partial_i A_j^{(1)} + [A_i^{(0)}, A_j^{(1)}]_*$, etc . Tr_d is the trace associated with the spacetime index i and TR corresponds to the trace of the different operators on the Hilbert space \mathbf{H} . As an illustrative example we compute now explicitly the quadratic effective action . This will also contain all quantum corrections to the vacuum polarization tensor . After a long calculation [3] one obtains

$$\Gamma_\theta^{(2)} = -\frac{1}{2} \int \frac{d^d \vec{p}}{(2\pi)^d} \left[(p^2 \delta_{ij} - p_i p_j) \left(\frac{1}{g^2} + \Pi^P(p) \right) + \Pi_{ij}^{NP}(p) \right] A_i^{(0)}(p) A_j^{(0)}(-p). \quad (7)$$

Explicitly we find in particular that the planar function is UV divergent as in the commutative theory and thus requires a renormalization. Indeed by integrating over arbitrarily high momenta in the internal loops we see that the planar amplitude diverges so at any arbitrary scale μ one finds in $d = 4 + 2\epsilon$ the closed expression [3]

$$\Pi^P(p) = -\frac{11}{24\pi^2} \left(\frac{1}{\epsilon} + \gamma + \ln \frac{p^2}{\mu^2} \right) + \frac{1}{24\pi^2} + \frac{1}{8\pi^2} \int_0^1 dx (1-2x)^2 \ln x (1-x) - \frac{1}{2\pi^2} \int_0^1 dx \ln x (1-x). \quad (8)$$

Obviously in the limit $\epsilon \rightarrow 0$ this planar amplitude diverges , i.e their singular high energy behaviour is logarithmically divergent . These divergent contributions needs therefore a renormalization . Towards this end it is enough as it turns out to add the following counter term to the bare action

$$\delta S_\theta = -\frac{1}{4} \left(\frac{11}{24\pi^2 \epsilon} \right) \int d^d x F_{ij}^{(0)2}. \quad (9)$$

The claim of [3, 4] is that this counter term will also subtract the UV divergences in the 3- and 4-point functions of the theory at one-loop and hence the theory is renormalizable at this order. The vacuum polarization tensor at one-loop is therefore given by

$$\Pi_{ij}^{1-loop} = (p^2 \delta_{ij} - p_i p_j) \frac{1}{g^2(\mu)} + \Pi_{ij}^{NP}(p). \quad (10)$$

where

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2} - \frac{11}{24\pi^2} \ln \frac{p^2}{\mu^2} - \frac{11}{24\pi^2} \gamma + \frac{1}{24\pi^2} + \frac{1}{8\pi^2} \int dx [(1-2x)^2 - 4] \ln x (1-x). \quad (11)$$

A straightforward calculation gives then the beta function

$$\beta(g(\mu)) = \mu \frac{dg(\mu)}{d\mu} = -\frac{11}{24\pi^2} g^3(\mu). \quad (12)$$

We remark on the other hand that the non-planar function $\Pi_{ij}^{NP}(p)$ is finite in the UV because of the presence of a regulating exponential of the form $\exp(-\frac{\tilde{p}^2}{4t})$ in loop integrals where $\tilde{p}_i = \theta^2 B_{ij} p_j$. However it is obvious that this noncommutativity-induced exponential regularizes the behaviour at high momenta (which corresponds to the values $t \rightarrow 0$) only when the external momentum \tilde{p} is $\neq 0$. Indeed in the limit of small noncommutativity or small momenta we have the infrared singular behaviour

$$\Pi_{ij}^{NP}(p) = \frac{11}{24\pi^2} (p^2 \delta_{ij} - p_i p_j) \ln p^2 \tilde{p}^2 + \frac{2}{\pi^2} \frac{\tilde{p}_i \tilde{p}_j}{(\tilde{p})^2}. \quad (13)$$

This also means that the renormalized vacuum polarization tensor diverges in the infrared limit $\tilde{p} \rightarrow 0$ which is the definition of the UV-IR mixing of this model.

In this article we will give a nonperturbative exact representation of the beta function (12) in the regime of strong noncommutativity using the method of large N matrix models. We will show in particular that the noncommutative $U(1)$ gauge theory is equivalent to an ordinary large non-linear sigma model and that the result (12) is actually valid to all orders in g . We postpone however the discussion of the UV-IR mixing problem (13) and its solution to a future communication [6].

2 The Fuzzy Sphere As a Regulator of The Moyal-Weyl Plane

As a warm up we will only consider in this section the case of two dimensions and then go through the 4-dimensional case in more detail in next sections. The action (1) reads in two dimensions as follows

$$S_\theta = \frac{\theta^2}{4g^2} Tr \hat{F}_{ij}^2 = \frac{\theta^2}{4g^2} Tr \sum_{i,j} \left(i[\hat{D}_i, \hat{D}_j] - \frac{1}{\theta^2} (\epsilon^{-1})_{ij} \right)^2. \quad (14)$$

The major obstacles in systematically quantizing the above action (14) are 1) the infinite dimensionality of the Fock space on which the trace Tr is defined, 2) the presence of a dimension-full parameter θ in the theory and 3) the absence of Lorentz invariance because of the existence of a background magnetic field B_{ij} [This last point is of course not relevant in the special case of 2 dimensions].

The above three problems are immediately solved by redefining the above action as certain limit of finite dimensional matrix models. Indeed in the case $d = 2$ (which is of most interest to us in this first section) we replace (14) by the $(L+1)$ -dimensional matrix model

$$S_{L,R} = \frac{R^2}{4g_f^2} \frac{1}{L+1} Tr_f F_{ab}^2 = \frac{R^2}{4g_f^2} \frac{1}{L+1} Tr_f \sum_{a,b} \left(i[D_a, D_b] + \sum_c \frac{1}{R} \epsilon_{abc} D_c \right)^2, \quad (15)$$

with the constraint [5, 7]

$$D_a D_a = \frac{|L|^2}{R^2}, \quad |L|^2 = \frac{L}{2} \left(\frac{L}{2} + 1 \right). \quad (16)$$

Now a, b, c take the values 1, 2, 3 which means that the above regularization is effectively embedded in 3 dimensions and hence the need for the extra constraint. The tensor ϵ_{abc} is the ϵ symbol in 3 dimensions. The trace Tr_f is now defined on a finite dimensional Hilbert space, this trace is dimensionless and the dimension of $(\text{length})^2$ which is carried by θ^2 in (14) is now carried by R^2 . The equations of motion derived from the action (15) are given by

$$\delta S_{L,R} = \frac{R^2}{g_f^2} \frac{1}{L+1} Tr_f \delta D_c \left(-i[F_{cb}, D_b] + \frac{1}{2R} \epsilon_{abc} F_{ab} \right) \equiv 0 \Leftrightarrow -i[F_{cb}, D_b] + \frac{i}{2R} \epsilon_{abc} F_{ab} = 0. \quad (17)$$

An important class of solutions to these equations of motion are given by the solutions to the zero-curvature condition $F_{ab} = 0$ together with the constraint (16). These are the famous so-called fuzzy spheres and they are essentially defined by the covariant derivatives $D_a = \frac{L_a}{R}$ for which $F_{ab} = 0$ and $D_a D_a = \frac{|L|^2}{R^2}$ where of course L_a 's are the generators of the $(L+1)$ -dimensional irreducible representation of $SU(2)$. It is also well established [8, 7, 11, 12] that these solutions are classically stable for finite L only because of the constraints (16) which we chose in here to impose rigidly [we could have instead chosen to implement these constraints in a variety of different ways as discussed in [7, 11]]. We replace therefore the configurations (2) by $(L+1) \times (L+1)$ matrices given by

$$D_a = \frac{1}{R} L_a + A_a. \quad (18)$$

The noncommutative coordinates \hat{x}_i 's are replaced by the noncommutative matrix coordinates $x_a = \frac{R}{|L|} L_a$'s satisfying

$$[x_a, x_b] = i \frac{R}{|L|} \epsilon_{abc} x_c, \quad \sum_a x_a^2 = R^2, \quad (19)$$

Hence we have effectively regularized the noncommutative plane (3) with a fuzzy sphere of radius R . This can also be seen as follows. We introduce the $(L+1) \times (L+1)$ gauge field and write $F_{ab} = F_{ab}^{(0)} + i[A_a, A_b]$, $F_{ab}^{(0)} = \frac{i}{R} ([L_a, A_b] - [L_b, A_a] - i\epsilon_{abc} A_c)$. The Yang-Mills action (15) in the large L limit becomes

$$S_{L,R} \longrightarrow \frac{R^2}{4g_f^2} \int \frac{d\Omega}{4\pi} (F_{ab}^{(0)})^2. \quad (20)$$

As one can immediately see this is indeed the $U(1)$ action on ordinary \mathbf{S}^2 with radius R and coupling constant g_f^2 .

However in the matrix model (15) we want to think of R and L as being infrared and ultraviolet cut-offs respectively of the theory (1) with the crucial property that for all finite values of these cut-offs gauge invariance and Lorentz invariance are preserved.

The limit in which the finite dimensional matrix model (15) reduces to the infinite dimensional matrix model (1) is a continuum double scaling limit of large R and large L taken as follows

$$R, L \longrightarrow \infty; \quad \text{keeping } \frac{R^2}{|L|^{2q}} = \text{fixed} \equiv \theta^2 \quad (21)$$

with q a real number and where we have also to constrain the fuzzy coordinate x_3 (for example via the application of an appropriate projector or by any other means) to be given by

$$x_3 = R.1. \quad (22)$$

This means that we are effectively restricting the theory around the north pole of the fuzzy sphere where in the limit of large R and large L one can reliably set $x_3 = R.1$. The noncommutative coordinates can then be identified as $\hat{x}_i = \frac{1}{|L|^{q-\frac{1}{2}}}(\epsilon^{-1})_{ij}x_j$ or $x_i = |L|^{q-\frac{1}{2}}\epsilon_{ij}\hat{x}_j$ with the correct commutation relations (3), i.e. $[\hat{x}_i, \hat{x}_j] = i\theta^2\epsilon_{ij}$. Furthermore by dividing R^2 across the identity $\sum_a x_a^2 = R^2$ one finds the trivial result $1 = 1$ which means in particular that the two coordinates \hat{x}_1, \hat{x}_2 are now not constrained in any way. The traces Tr_f and Tr are on the other hand identified in the planar limit through the simple equation $Tr = Tr_f$ [9]. Furthermore in this large planar limit (21) the constraint (16) takes the form

$$\frac{1}{R}\{x_3, A_3\} + \frac{1}{R}\sum_{i=1}^2\{x_i, A_i\} + \theta|L|^{q-1}\sum_{a=1}^3A_a^2 = 0 \Leftrightarrow \frac{2}{\theta}\hat{A}_3 + \frac{1}{|L|}\hat{A}_3^2 + \frac{1}{\theta^2|L|}\{\hat{x}_i, \hat{A}_i\} + \frac{1}{|L|}\hat{A}_i^2 = 0. \quad (23)$$

In other words $\hat{A}_3 = 0$ in this limit and thus one finds $\hat{D}_3 = \frac{1}{\theta}$ where $\hat{A}_3 = |L|^{q-1}A_3$ and $\hat{D}_3 = |L|^{q-1}D_3$. Clearly we are also using the fact that we have in this limit $\hat{D}_i = |L|^{q-\frac{1}{2}}D_i$ and $\hat{A}_i = |L|^{q-\frac{1}{2}}A_i$ where of course $\hat{D}_i = -\frac{1}{\theta^2}(\epsilon^{-1})_{ij}\hat{x}_j + \hat{A}_i$. As a consequence we conclude that $F_{ij} = \frac{1}{|L|^{\frac{1}{2q-1}}}\hat{F}_{ij}$ where $\hat{F}_{ij} = i[\hat{D}_i, \hat{D}_j] - \frac{1}{\theta^2}(\epsilon^{-1})_{ij}$ and $F_{i3} = -\frac{1}{\theta|L|^{2q-\frac{1}{2}}}\epsilon_{ij}\hat{D}_j$. The sum \sum_{ab} reduces effectively to \sum_{ij} , i.e. the difference $S_{L,R} - \frac{1}{L+1}\frac{1}{|L|^{\frac{1}{2q-2}}}S_\theta$ being equal to $\frac{1}{2g_f^2|L|^{2q-1}(L+1)}Tr\sum_i(\hat{D}_i)^2$ consists only of vanishing terms of order $\frac{1}{L^{2q}}$ and hence the action (15) is seen to tend to (14) with an effective classical coupling given by

$$g_{eff}^2 = g_f^2\xi^2, \quad \xi^2 = |L|^{2q-2}(L+1). \quad (24)$$

However and since we have

$$S_\theta = \frac{\theta^2}{4g_{eff}^2}Tr\left([\hat{D}_i, \hat{D}_j] - \frac{1}{\theta^2}(\epsilon^{-1})_{ij}\right)^2 = \frac{\theta_{eff}^2}{4g_f^2}Tr\left([\frac{1}{\xi}\hat{D}_i, \frac{1}{\xi}\hat{D}_j] - \frac{1}{\theta_{eff}^2}(\epsilon^{-1})_{ij}\right)^2, \quad (25)$$

we can view (15) as describing (in the limit) a gauge theory on a noncommutative plane with an effective noncommutativity parameter given by

$$\theta_{eff}^2 = \theta^2\xi^2. \quad (26)$$

From here we can conclude that for $q > \frac{1}{2}$, $\xi^2 \rightarrow \infty$ when $L \rightarrow \infty$ and thus θ_{eff} corresponds to strong noncommutativity. For $q < \frac{1}{2}$ we find that $\xi^2 \rightarrow 0$ when $L \rightarrow \infty$ and θ_{eff} corresponds to weak noncommutativity whereas for $q = \frac{1}{2}$ the effective noncommutativity parameter is exactly given by $\theta_{eff}^2 = 2\theta^2$. Let us point out here that the above result can also be derived from coherent states and star products.

We can immediately see from (15) that Lorentz invariance is here fully maintained at the level of the action in the form of the explicit rotational $SU(2)$ symmetry of the fuzzy sphere. The $SO(3)$ symmetry is broken down to $SO(2)$ symmetry only by the constraint (22). Furthermore the noncommutativity parameter θ^2 from (19), (21) and (22) provides the only length scale in the problem and hence θ for all values of R and L defines the volume and distances of the underlying space-time and therefore it can not be treated as some dimensionfull coupling constant in the theory.

3 The Chern-Simons Action

It was shown in [8] that the dynamics of open strings moving in a curved space with S^3 metric in the presence of a non-vanishing Neveu-schwarz B-field and with Dp-branes is not precisely equivalent, to the leading order in the string tension, to the above gauge theory (15). This is of course in contrast with the case of strings in flat backgrounds. Indeed the effective action turns out to contain also an extra crucial term given by the Chern-Simons action

$$S_{CS} = -\frac{R}{6g_f^2}\frac{1}{L+1}\epsilon_{abc}Tr_f F_{ab}D_c - \frac{1}{6g_f^2}\frac{1}{L+1}Tr_f(D_a^2 - \frac{|L|^2}{R^2}). \quad (27)$$

From string theory point of view the most natural candidate for a gauge action on the fuzzy sphere is therefore given instead by the action

$$S_L = S_{L,R} + S_{CS}. \quad (28)$$

We remark that the Chern-Simons term vanishes in the planar limit (21) and thus its addition does not change the argument of the previous section. This fact can also be seen by rewriting the Chern-Simons action in terms of the gauge field directly as follows. We write $D_a = \frac{1}{R}L_a + A_a$ and then compute

$$S_{CS} = -\frac{1}{2g_f^2} \frac{1}{L+1} \epsilon_{abc} \text{Tr}_f \left[\frac{1}{2} F_{ab}^{(0)} A_c + \frac{iR}{3} [A_a, A_b] A_c \right] = -\frac{1}{2g_f^2} \frac{1}{L+1} \text{Tr}_f \left[\frac{1}{2} \epsilon_{ij3} F_{ij}^{(0)} A_3 + R \epsilon_{3ij} A_j F_{3i} \right] \quad (29)$$

where $F_{ab}^{(0)} = i[L_a, A_b] - i[L_b, A_a] + \epsilon_{abc} A_c$, $F_{ab} = \frac{1}{R} F_{ab}^{(0)} + i[A_a, A_b]$. Hence in the planar limit where we can set $A_3 = 0$, $A_i = \frac{1}{|L|^{q-\frac{1}{2}}} \hat{A}_i$ and $F_{3i} = \frac{1}{\theta |L|^{2q-\frac{1}{2}}} \epsilon_{ij} \hat{D}_j$ it is quite obvious that we will have $S_{CS} = -\frac{1}{2g_f^2} \frac{1}{L+1} \text{Tr}_f \frac{\hat{A}_i \hat{D}_i}{|L|^{2q-1}}$, i.e this action vanishes also as $\frac{1}{L^{2q}}$.

As it turns out however the addition of the Chern-Simons term simplifies considerably perturbation theory. Indeed one can check that the quadratic term of the action S_L is of the form

$$S_L^{(2)} = -\frac{1}{2g_f^2} \frac{1}{L+1} \text{Tr}_f ([L_a, A_b]^2 - [L_a, A_a]^2). \quad (30)$$

In other words and after an obvious gauge fixing the propagator of the theory is simply given by $\frac{g_f^2}{L^2}$ which is very similar to the propagator on the plane. This simplification seems to be related to the fact that the action S_L has the extra symmetry $A_a \rightarrow A_a + \alpha_a \mathbf{1}_{L+1}$ for any constants α_a , in other words it is invariant under global translations in the space of gauge fields. We choose for simplicity to fix this symmetry by restricting the gauge field to be traceless, i.e by removing the zero modes. The action we will study is therefore given by S_L with the constraint (16) and the corresponding partition function is defined by

$$Z_L[g_f^2 \theta^2; J] = \int \prod_{a=1}^3 \mathcal{D}D_a \delta \left(D_a^2 - \frac{|L|^2}{R^2} \right) e^{-S_L - \frac{R}{L+1} \text{Tr}_f J_a D_a}. \quad (31)$$

This theory was extensively studied for finite L (keeping R fixed) in [7]. As we have said earlier the constraint $D_a^2 - \frac{|L|^2}{R^2} = 0$ simply removes the normal component of the gauge field which is defined here by $\Phi = \frac{D_a^2 - |L|^2}{2|L|}$. In [7] we have shown explicitly that without this constraint the model (28) has a gauge-invariant UV-IR mixing. Furthermore by adding a large mass term for the normal component of the gauge field in the form $M^2 \text{Tr}_f \Phi^2$ we can show that, in the limit where $M \rightarrow \infty$ first (which will implement the constraint) then $L \rightarrow \infty$, the mixing is removed. This result is confirmed by the large L analysis of [11] and suggests that the UV-IR mixing has its origin in the coupling of extra degrees of freedom to the theory which are here identified with the scalar normal component of the gauge field. The other exciting result regarding this model is the existence of a first order phase transition in the system at some large coupling between a pure matrix model and a fuzzy sphere model. This phase transition was confirmed numerically by [12] and suggests that the one-loop quantum theory is actually an exact result.

4 $U_*(1)$ Theory in d -Dimensions

The space \mathbf{R}_θ^d in general can be only partially noncommutative, i.e the Poisson tensor $\theta^2 B_{ij}$ is of rank $2r \leq d$. This means in particular that we have only $2r$ noncommuting coordinates. We will now concentrate on the case of $U(1)$ gauge theory on a minimal noncommutative space, i.e $r = 1$. The notation for $i = 1, 2$ remains \hat{x}_i which correspond in the star picture to the noncommutative coordinates x_1 and x_2 (or equivalently the complex coordinates $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$). For $i = 3, \dots, d$ or $\mu = 1, \dots, d-2$ we have the commutative coordinates $\hat{x}_i \equiv x_\mu$. The commutation relations are therefore

$$[\hat{x}_i, \hat{x}_j] = i\theta^2 \epsilon_{ij}, \quad [x_\mu, x_\nu] = [\hat{x}_i, x_\mu] = 0, \quad (32)$$

where we have set $B_{12} \equiv \epsilon_{12} = 1$ for simplicity [1]. The derivatives on this noncommutative space will now be defined by

$$\begin{aligned} \hat{\partial}_i &= -\frac{i}{\theta^2} (\epsilon^{-1})_{ij} \hat{x}_j = -\hat{\partial}_i^+, \quad [\hat{\partial}_i, \hat{\partial}_j] = \frac{i}{\theta^2} (\epsilon^{-1})_{ij} = -\frac{i}{\theta^2} \epsilon_{ij}, \quad [\hat{\partial}_i, x_\mu] = 0, \quad (\text{for } i = 1, 2) \\ \hat{\partial}_i &\equiv \partial_\mu = -\partial_\mu^+, \quad [\partial_\mu, \hat{x}_i] = 0, \quad [\partial_\mu, x_\nu] = \delta_{\mu\nu} \quad (\text{for } i = 3, \dots, d, \quad \mu = 1, \dots, d-2). \end{aligned} \quad (33)$$

Also we have $[\partial_\mu, \hat{\partial}_i] = 0$, $i = 1, 2$. The covariant derivatives are on the other hand given by

$$\begin{aligned} \hat{D}_i &= -i\hat{\partial}_i + \hat{A}_i, \quad (\text{for } i = 1, 2) \\ \hat{D}_i &\equiv \hat{D}_\mu = -i\hat{\partial}_i + \hat{A}_i \equiv -i\partial_\mu + \hat{A}_\mu, \quad (\text{for } i = 3, \dots, d, \quad \mu = 1, \dots, d-2). \end{aligned} \quad (34)$$

Both \hat{A}_i and \hat{A}_μ are still operators, indeed we can write the Fourier expansion

$$\hat{A}_i \equiv \hat{A}_i(\hat{x}_1, \hat{x}_2, x_\mu) = \int \frac{d^d k}{(2\pi)^d} \tilde{A}_i(k) e^{ik_1 \hat{x}_1 + ik_2 \hat{x}_2} e^{ik_\mu x_\mu}, \quad (\text{for all } i = 1, \dots, d). \quad (35)$$

The operators \hat{A}_i 's clearly act on the same Hilbert space \mathbf{H} on which the coordinate operators \hat{x}_1 and \hat{x}_2 act. The operators \hat{A}_i 's can be mapped to the fields A_i given by

$$\hat{A}_i(\hat{x}_1, \hat{x}_2, x_\mu) = \int d^2 x A_i(x_1, x_2, x_\mu) \Delta(\hat{x}_1, \hat{x}_2, x_1, x_2), \quad (36)$$

where the Weyl map is given by

$$\Delta(\hat{x}_1, \hat{x}_2, x_1, x_2) = \int \frac{d^2 k}{(2\pi)^2} e^{ik_i \hat{x}_i} e^{-ik_i x_i}. \quad (37)$$

Remark for example that if \hat{A}_i did not depend on the operators \hat{x}_1 and \hat{x}_2 then one can simply make the identification $\hat{A}_i(x_\mu) \equiv A_i(x_\mu)$ since $\int d^2 x \Delta(\hat{x}_1, \hat{x}_2, x_1, x_2) = 1$. Indeed the star product is given now by

$$f * g(x) = e^{\frac{i}{2}\theta^2 \sum_{i,j=1}^2 \epsilon_{ij} \frac{\partial}{\partial \hat{x}_i} \frac{\partial}{\partial \hat{x}_j}} f(x + \xi) g(x + \eta) |_{\xi=\eta=0}, \quad (38)$$

and clearly it involves only the two derivatives $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ so if both f and g do not depend on the two coordinates x_1 and x_2 then $f * g(x) \equiv f(x)g(x)$. In fact even in the case where only one of the two functions f and g is independent of x_1 and x_2 we have $f * g(x) \equiv f(x)g(x)$.

The curvature is defined now by

$$\begin{aligned} \hat{F}_{ij} &= i[\hat{D}_i, \hat{D}_j] + \frac{1}{\theta^2} \epsilon_{ij} = [\hat{\partial}_i, \hat{A}_j] - [\hat{\partial}_j, \hat{A}_i] + i[\hat{A}_i, \hat{A}_j] \\ \hat{F}_{\mu i} &= i[\hat{D}_\mu, \hat{D}_i] = \partial_\mu \hat{A}_i - [\hat{\partial}_i, \hat{A}_\mu] + i[\hat{A}_\mu, \hat{A}_i] \\ \hat{F}_{\mu\nu} &= i[\hat{D}_\mu, \hat{D}_\nu] = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + i[\hat{A}_\mu, \hat{A}_\nu], \end{aligned} \quad (39)$$

where i above stands for the two values 1 and 2 and μ stands for the rest. Gauge transformations are also operators \hat{U} which act as usual, namely

$$\begin{aligned} \hat{D}_\mu^U &= \hat{U} \hat{D}_\mu \hat{U}^+ \longrightarrow \hat{A}_\mu^U = \hat{U} \hat{A}_\mu \hat{U}^+ - i\hat{U} \partial_\mu (\hat{U}^+) \\ \hat{D}_i^U &= \hat{U} \hat{D}_i \hat{U}^+ \longrightarrow \hat{A}_i^U = \hat{U} \hat{A}_i \hat{U}^+ - i\hat{U} [\hat{\partial}_i, \hat{U}^+], \end{aligned}$$

and hence $\hat{F}_{ij}^U = \hat{U} \hat{F}_{ij} \hat{U}^+$ for all $i, j = 1, \dots, d$. The Yang-Mills action for $U_\star(1)$ gauge theory on $\mathbf{R}_\theta^2 \times \mathbf{R}^{d-2}$ is written in this case as

$$\begin{aligned} S_\theta &= \frac{\theta^2}{4g^2} \int d^{d-2}x \sum_{i,j=1}^d \text{Tr} \hat{F}_{ij}^2 \\ &= \frac{\theta^2}{4g^2} \int d^{d-2}x \text{Tr} \hat{F}_{\mu\nu}^2 + \frac{\theta^2}{2g^2} \int d^{d-2}x \sum_{i=1}^2 \text{Tr} \hat{F}_{\mu i}^2 + \frac{\theta^2}{4g^2} \int d^{d-2}x \sum_{i,j=1}^2 \text{Tr} \hat{F}_{ij}^2. \end{aligned} \quad (40)$$

In above we have deliberately used the fact that we can replace the integral over the noncommutative directions x_1 and x_2 by a trace over an infinite dimensional Hilbert space by using the Weyl Map introduced in (37). By doing this we have therefore also replaced the underlying star product of functions by pointwise multiplication of operators. The trace Tr in (40) is thus associated with the two noncommutative coordinates x_1 and x_2 . It is curious enough however that the above model looks very much like a $U(\infty)$ gauge theory on \mathbf{R}^{d-2} with a Higgs particle in the adjoint of the group. This is in fact our original motivation for wanting to regularize the NC plane with a fuzzy sphere.

For each point x_μ of the $(d-2)$ -dimensional commutative submanifold \mathbf{R}^{d-2} , the action (40) is essentially an infinite dimensional matrix model and hence it can be regularized and made into a finite dimensional matrix model if we approximate for example the noncommutative plane by a fuzzy sphere. As we explained earlier the trace $\frac{\theta^2}{g^2} \text{Tr}$ will be replaced by $\frac{R^2}{g_L^2} \text{Tr}_f$ where $g_L^2 = g_f^2(L+1)$ and the first two terms in the action become

$$\frac{\theta^2}{4g^2} \int d^{d-2}x \text{Tr} \hat{F}_{\mu\nu}^2 + \frac{\theta^2}{2g^2} \int d^{d-2}x \sum_{i=1}^2 \text{Tr} \hat{F}_{\mu i}^2 \longrightarrow \frac{1}{4\lambda^2} \int d^{d-2}x \text{Tr}_f \mathcal{F}_{\mu\nu}^2 - \frac{1}{2\lambda^2} \int d^{d-2}x \sum_{a=1}^3 \text{Tr}_f [\mathcal{D}_\mu, D_a]^2, \quad (41)$$

with $\lambda^2 = \frac{g_L^2}{R^2} = \frac{g_f^2(L+1)}{R^2}$, where we have also replaced the operators $\hat{A}_\mu = \hat{D}_\mu + i\partial_\mu$ and \hat{D}_i by the $(L+1) \times (L+1)$ dimensional matrices $\mathcal{A}_\mu = \mathcal{D}_\mu + i\partial_\mu$ and D_a respectively. In above $\mathcal{F}_{\mu\nu} = i[\mathcal{D}_\mu, \mathcal{D}_\nu]$ while the index a runs over 1, 2, 3 since the fuzzy sphere is described by a 3-dimensional calculus. In here the fuzzy sphere is only thought of as a regulator of the noncommutative plane which preserves exact gauge invariance. Classically we have found that $g_f^2 = g^2$ whereas the effective noncommutativity parameter appearing in the Moyal-Weyl action is $\theta^2 \xi^2$. The coupling constant λ^2 has dimension M^{6-d} whereas the covariant derivatives \mathcal{D}_μ and D_a are as before of dimension M . Remark furthermore that in the continuum planar limit $L, R \rightarrow \infty$ keeping θ fixed in which the fuzzy sphere reproduces the noncommutative plane the combination $\lambda^2 \frac{|L|^{2q}}{L+1}$ is kept fixed equal to $\frac{g_f^2}{\theta^2}$: this is the analogue of 't Hooft planar limit in this context.

The last term in (40) has the following interpretation. For each point x_μ of the $(d-2)$ -dimensional commutative submanifold this term is exactly equivalent to a $U(1)$ gauge theory on a noncommutative \mathbf{R}^2 . This term and as we have explained in previous sections will therefore be regularized by the sum of the actions (15) +(27). In terms of D_a this action reads

$$\frac{\theta^2}{4g^2} \int d^{d-2}x \sum_{i,j=1}^2 \text{Tr} \hat{F}_{ij}^2 \longrightarrow -\frac{R^2}{4g_L^2} \int d^{d-2}x V(D_a), \quad (42)$$

where

$$V(D_a) = \text{Tr}_f [D_a, D_b]^2 - \frac{4i}{3R} \epsilon_{abc} \text{Tr}_f [D_a, D_b] D_c - \frac{2}{3R^4} (L+1) |L|^2. \quad (43)$$

As opposed to the case of perturbation theory where the Chern-Simons term played a crucial role in simplifying the propagator and as a consequence the model as a whole, we can see in here that in the large R limit the Chern-Simons contribution is rather small compared to the Yang-Mills contribution and hence this term becomes irrelevant in this limit.

The full regularized action $S_{\theta,L}$ becomes

$$S_{\theta,L} = \frac{1}{4\lambda^2} \int d^{d-2}x \text{Tr}_f \mathcal{F}_{\mu\nu}^2 - \frac{1}{2\lambda^2} \int d^{d-2}x \sum_{a=1}^3 \text{Tr}_f [\mathcal{D}_\mu, D_a]^2 - \frac{1}{4\lambda^2} \int d^{d-2}x V(D_a). \quad (44)$$

The matrices \mathcal{A}_μ , D_a are of course still functions on the commutative part \mathbf{R}^{d-2} of \mathbf{R}_θ^d , i.e $\mathcal{A}_\mu \equiv \mathcal{A}_\mu(x_\mu)$, $D_a \equiv D_a(x_\mu)$, $\mu = 1, 2, \dots, d-2$. Gauge transformations $\hat{U} \equiv \hat{U}(x_\mu)$ of the original action are implemented now by unitary transformations $U \equiv U(x_\mu)$ acting on the $(L+1)$ -dimensional Hilbert space of the irreducible representation $\frac{L}{2}$ of $SU(2)$ which leave the above action $S_{\theta,L}$ invariant. They are given explicitly by $D_a \rightarrow U D_a U^\dagger$, $\mathcal{D}_\mu \rightarrow U \mathcal{D}_\mu U^\dagger$ (or equivalently $\mathcal{A}_\mu \rightarrow U \mathcal{A}_\mu U^\dagger - iU \partial_\mu U^\dagger$ where $\mathcal{D}_\mu = -i\partial_\mu + \mathcal{A}_\mu$). This is clearly a $U(L+1)$ gauge theory with adjoint matter, i.e the original noncommutative degrees of freedom are traded for ordinary color degrees of freedom which in fact resembles very much what happens on the noncommutative torus under Morita equivalence. In particular the components of the covariant derivative in the noncommutative directions, i.e D_a , $a = 1, 2, 3$, are now simple scalar fields with respects to the other commutative $d-2$ dimensions. Quantization of the above model with the constraint (16) corresponds therefore to an ordinary quantum field theory.

4.1 The Non-Linear Sigma Model

4.1.1 Light-Cone Gauge

We use now the notation $N \equiv L+1$ and work in $d=4$. The field \mathcal{A}_μ can be separated into a $U(1)$ gauge field a_μ and an $SU(N)$ gauge field A_μ as follows

$$\mathcal{A}_\mu(x) = a_\mu(x)1 + A_\mu(x), \quad A_\mu(x) = A_{\mu A}(x)T_A, \quad (45)$$

where we have introduced the $SU(N)$ -Gell-Mann matrices $T_A = \frac{\lambda_A}{2}$, $A = 1, \dots, N^2-1$, which satisfy the usual conditions

$$T_A^\dagger = T_A, \quad \text{Tr}_f T_A = 0, \quad \text{Tr}_f T_A T_B = \frac{1}{2} \delta_{AB}, \quad [T_A, T_B] = i f_{ABC} T_C. \quad (46)$$

The curvature becomes

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= f_{\mu\nu} + F_{\mu\nu} \\ f_{\mu\nu} &= \partial_\mu a_\nu - \partial_\nu a_\mu \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] = F_{\mu\nu C} T_C, \quad F_{\mu\nu C} = \partial_\mu A_{\nu C} - \partial_\nu A_{\mu C} - f_{ABC} A_{\mu A} A_{\nu B}. \end{aligned} \quad (47)$$

The first term in the action becomes

$$\frac{1}{4\lambda^2} \int d^{d-2}x \text{Tr}_f \mathcal{F}_{\mu\nu}^2 = \frac{1}{4\lambda^2} \int d^{d-2}x \left[N f_{\mu\nu}^2 + \text{Tr}_f F_{\mu\nu}^2 \right]. \quad (48)$$

Similarly the gauge transformation $U = \exp(i\Lambda(x))$, where Λ is a general $N \times N$ matrix, splits into a $U(1)$ gauge transformation and an $SU(N)$ gauge transformation, i.e $U \equiv e^{i\Lambda(x)} = e^{i(\alpha(x) + \beta(x))} = W(x) \cdot V(x)$ where $W(x) = e^{i\alpha(x)}$ and $V(x) = e^{i\beta(x)}$ with $\alpha(x)$ a function on \mathbf{R}^{d-2} and $\beta(x) = \beta_A(x)T_A$. Hence the gauge transformation $\mathcal{A}_\mu \rightarrow \mathcal{A}_\mu^U = U \mathcal{A}_\mu U^\dagger - iU \partial_\mu U^\dagger$ takes now the form

$$a_\mu \rightarrow a_\mu^U = a_\mu - iW(\partial_\mu W^\dagger), \quad A_\mu \rightarrow A_\mu^U = V A_\mu V^\dagger - iV(\partial_\mu V^\dagger). \quad (49)$$

Similarly we write

$$D_a = n_a + \Phi_a, \quad \Phi_a = \Phi_{aA} T_A, \quad (50)$$

In this case the gauge transformation $D_a \rightarrow D_a^U = U D_a U^\dagger$ becomes

$$n_a \rightarrow n_a^U = n_a, \quad \Phi_a \rightarrow \Phi_a^U = V \Phi_a V^\dagger. \quad (51)$$

This means that Φ_a is a scalar field ("scalar" with respect to the commutative directions of \mathbf{R}_θ^d) which transforms in the adjoint representation of the non-abelian subgroup $SU(N)$ of $U(N)$. In fact n_a is also a scalar field in the same sense. The action takes now the explicit form

$$\begin{aligned} S_{\theta;L} = & \frac{1}{4\lambda^2} \int d^{d-2} x \text{Tr}_f F_{\mu\nu}^2 + \frac{N}{4\lambda^2} \int d^{d-2} x f_{\mu\nu}^2 - \frac{1}{2\lambda^2} \int d^{d-2} x \text{Tr}_f [D_\mu, \Phi_a]^2 + \frac{N}{2\lambda^2} \int d^{d-2} x (\partial_\mu n_a)^2 \\ & - \frac{1}{4\lambda^2} \int d^{d-2} x V(\Phi_a), \end{aligned} \quad (52)$$

where $D_\mu = -i\partial_\mu + A_\mu$ and $V(\Phi_a) = \text{Tr}_f [\Phi_a, \Phi_b]^2 - \frac{4i}{3R} \epsilon_{abc} \text{Tr}_f [\Phi_a, \Phi_b] \Phi_c - \frac{2}{3\theta^4} \frac{L+1}{|L|^2}$. The second term in this action is trivial describing an abelian $U(1)$ gauge field on an Euclidean $(d-2)$ -dimensional flat spacetime with no interactions with the other fields. In the case of $d=4$ the non-abelian part of the action (i.e. the first term in (52)) is seen to be defined on a two dimensional spacetime and thus it can be simplified further if one uses the light-cone gauge [10]. To this end we rotate first to Minkowski signature then we fix the $SU(L+1)$ symmetry by going to the light-cone gauge given by $A_1 = A_2 = \sqrt{2}\lambda A_+$ (this is equivalent to $A_- = 0$). Similarly we fix the $U(1)$ gauge symmetry by writing $a_\mu = \sqrt{\frac{2}{N}} \lambda (\epsilon_{\mu\lambda} \partial^\lambda \sigma + \partial_\mu \eta)$. The action becomes therefore

$$\begin{aligned} \exp(iS_{\theta;L}) = & \exp i \left(- \int d^2 x (\partial^2 \sigma)^2 - \int d^2 x (\partial_- A_+)^2 + \frac{N}{2\lambda^2} \int d^2 x (\partial_\mu n_a)(\partial^\mu n_a) + \frac{1}{4\lambda^2} \int d^2 x (\partial_\mu \Phi_{aA})(\partial^\mu \Phi_{aA}) \right. \\ & \left. - \frac{1}{\lambda} f_{ABC} \int d^2 x (\partial_- \Phi_{aA}) A_{+B} \Phi_{aC} - \frac{1}{4\lambda^2} \int d^2 x V(\Phi_a) \right). \end{aligned} \quad (53)$$

Remark from above that there is no ghost term in the light-cone gauge [10]. The partition function is of the form

$$Z = \int \mathcal{D}\sigma \mathcal{D}A_+ \mathcal{D}n_a \mathcal{D}\Phi_{aA} e^{iS_{\theta;L}} \delta(D_a^2 - \frac{|L|^2}{R^2}). \quad (54)$$

The delta function is clearly inserted in order to implement the constraint (16). It is rather trivial to see that the field σ is completely decoupled from the rest of the dynamics and so it simply drops out from the action whereas we notice that we can perform the integral over the A_+ fields in a straightforward manner to give a non-local Coulomb interaction between the Φ_{aC} fields. We define $f_{ABC}(\partial_- \Phi_{aA}) \Phi_{aC} \equiv (\vec{\Phi}_a \times_L \partial_- \vec{\Phi}_a)_B$ and then write the final result in the form

$$\begin{aligned} \hat{S}_{\theta;L} = & \frac{N}{2\lambda^2} \int d^2 x (\partial_\mu n_a)(\partial^\mu n_a) + \frac{1}{4\lambda^2} \int d^2 x (\partial_\mu \Phi_{aA})(\partial^\mu \Phi_{aA}) \\ & - \frac{1}{4\lambda^2} \int d^2 x d^2 y (\vec{\Phi}_a \times_L \partial_- \vec{\Phi}_a)_A(x) D_{AB}^{-1}(x, y) (\vec{\Phi}_b \times_L \partial_- \vec{\Phi}_b)_B(y) - \frac{1}{4\lambda^2} \int d^2 x V(\Phi_a). \end{aligned} \quad (55)$$

$D_{AB}^{-1}(x, y)$ is the propagator of the A_+ fields, i.e. $D_{AB}^{-1}(x, y) = -\frac{\delta_{AB}}{2\pi} |x_- - y_-| \delta(x_+ - y_+)$.

4.1.2 The Constraint

Next we analyze the constraint $D_a D_a = \frac{|L|^2}{R^2}$. This can be rewritten in the form

$$n_a^2 + \frac{1}{2N} \Phi_{aA}^2 = \frac{|L|^2}{R^2}, \quad n_a \Phi_{aC} + \frac{1}{4} d_{ABC} \Phi_{aA} \Phi_{aB} = 0, \quad (56)$$

where we have used the identities $T_A T_B = \frac{1}{2N} \delta_{AB} + \frac{1}{2} (d_{ABC} + i f_{ABC}) T_C$, $\text{Tr} T_A T_B T_C = \frac{1}{4} (d_{ABC} + i f_{ABC})$. From the structure of this constraint and from the action (55) we can see that the field n_a appears at most

quadratically and hence it can be integrated out without much effort . The relevant part of the partition function reads

$$\begin{aligned} Z_{\vec{n}} &= \int \mathcal{D}n_a \exp\left(-\frac{N}{2\lambda^2} \int d^2x (\partial_\mu n_a)^2\right) \delta\left(n_a^2 + \frac{1}{2N} \Phi_{aA}^2 - \frac{1}{\theta^2}\right) \delta\left(n_a \Phi_{aC} + \frac{1}{4} d_{ABC} \Phi_{aA} \Phi_{aB}\right) \\ &= \int \mathcal{D}n_a \mathcal{D}J \mathcal{D}J_C \exp\left(-\frac{N}{2\lambda^2} \int d^2x (\partial_\mu n_a)^2 + iJ\left(n_a^2 + \frac{1}{2N} \Phi_{aA}^2 - \frac{|L|^2}{R^2}\right) + iJ_C\left(n_a \Phi_{aC} + \frac{1}{4} d_{ABC} \Phi_{aA} \Phi_{aB}\right)\right). \end{aligned} \quad (57)$$

The delta functions which are obviously enforcing the constraint are represented for convenience with Lagrange multiplier fields J and J_C . In the above partition function $Z_{\vec{n}}$ we have also rotated back to Euclidean signature for ease of manipulations . The equations of motion read as follows

$$\partial^2 n_a = -\frac{2i\lambda^2}{N} (J n_a + \frac{1}{2} J_C \Phi_{aC}). \quad (58)$$

Writing now $n_a = e_a + q_a$ where the fixed background field e_a is assumed to solve the above equations of motion whereas q_a is the fluctuation field one can compute in a straightforward manner the partition function

$$Z_{\vec{n}} = \delta\left(\frac{1}{2} e_a \Phi_{aC} + \frac{1}{4} d_{ABC} \Phi_{aA} \Phi_{aB}\right) \int \mathcal{D}J \exp\left(\frac{3}{2} \int d^2x < x | \text{Log}[\partial^2 + \frac{2i\lambda^2}{N} J] | x > + i \int d^2x J \left(\frac{1}{2N} \Phi_{aA}^2 - \frac{|L|^2}{R^2}\right)\right). \quad (59)$$

In the large L limit the exact quantum result $\frac{3}{2} \int d^2x < x | \text{Log}[\partial^2 + \frac{2i\lambda^2}{N} J] | x >$ becomes independent of J and hence the above partition function reduces simply to a product of two delta functions , namely $Z_{\vec{n}} = \delta\left(\frac{1}{2} e_a \Phi_{aC} + \frac{1}{4} d_{ABC} \Phi_{aA} \Phi_{aB}\right) \delta\left(\frac{1}{2N} \Phi_{aA}^2 - \frac{|L|^2}{R^2}\right)$ where now e_a is the solution of the equation $\partial^2 e_a \rightarrow 0$. In other words the integration over the field n_a in the large L limit is essentially equivalent to imposing on the field $\chi_{aA} = \frac{R}{|L|} \frac{1}{\sqrt{2N}} \Phi_{aA}$ the following constraint

$$\chi_{aA}^2 = 1, \quad d_{ABC} \chi_{aA} \chi_{aB} = -\frac{2e_a R}{|L| \sqrt{2(L+1)}} \chi_{aC}. \quad (60)$$

From the above derivation this result clearly does not depend on the metric we used and so it must also be valid for Minkowski signature . Since in the limit the vector e_a is an arbitrary solution of $\partial^2 e_a = 0$ we take it for simplicity x -independent . The reduced action becomes on the other hand

$$\bar{S}_{\theta;L} = \frac{1}{4\bar{\lambda}^2} \int d^2x (\partial_\mu \chi_{aA}) (\partial^\mu \chi_{aA}) - \frac{|L|^2(L+1)}{2\lambda^2 R^2} \int d^2x \bar{V}(\chi_a), \quad (61)$$

where

$$\begin{aligned} \bar{V}(\chi_a) &= \int d^2y (\vec{\chi}_a \times_L \partial_- \vec{\chi}_a)_A(x) D_{AB}^{-1}(x, y) (\vec{\chi}_b \times_L \partial_- \vec{\chi}_b)_B(y) + \text{Tr}_f [\chi_a, \chi_b]^2 - \frac{4i}{3|L| \sqrt{2(L+1)}} \epsilon_{abc} \text{Tr}_f [\chi_a, \chi_b] \chi_c \\ &- \frac{1}{6|L|^2(L+1)}. \end{aligned} \quad (62)$$

In here $\bar{\lambda}^2 = \frac{g_f^2}{2|L|^2}$. Since $R^2 = \theta^2 |L|^{2q}$ the coupling in front of the potential \bar{V} behaves in the limit as $\frac{|L|^2(L+1)}{2\lambda^2 R^2} \sim \frac{1}{\lambda^2 \theta^2} \left(\frac{L}{2}\right)^{3-2q}$ and thus for all scalings with $q > \frac{3}{2}$ this potential term can be neglected compared to the kinetic term and one ends up with the following partition function

$$Z = \int \mathcal{D}\chi_{aA} \delta(\chi_{aA}^2 - 1) \delta\left(d_{ABC} \chi_{aA} \chi_{aB} + 2\theta |L|^{q-\frac{3}{2}} e_a \chi_{aA}\right) e^{-\bar{S}_{\theta;L}}. \quad (63)$$

As we have discussed earlier the fuzzy theory for these particular scalings becomes a theory living on a noncommutative plane with effective deformation parameter given by $\theta_{eff}^2 \sim 2\theta^2 \left(\frac{L}{2}\right)^{2q-1}$ (see equation (26)) . We are

therefore probing the strong noncommutativity region of the Moyal-Weyl model . The above partition function can be easily computed and one finds

$$Z = \int \mathcal{D}J \mathcal{D}J_C e^{i \int d^2x J} \exp\left(-\frac{3}{2} T R \log D\right) \exp\left(-\epsilon^2 \theta^2 |L|^{2(q-\frac{3}{2})} \int d^2x d^2y J_A(x) D_{AB}^{-1}(x, y) J_B(y)\right), \quad (64)$$

where $D(= D_{AB}(x, y))$ is now the Laplacian

$$D_{AB}(x, y) = \delta^2(x - y) \left(-\frac{1}{4\lambda^2} \partial^2 \delta_{AB} + iJ \delta_{AB} + iJ_C d_{ABC} \right). \quad (65)$$

At this stage it is obvious that in the large L limit only configurations where $J_A = 0$ are relevant and thus one ends up with the partition function

$$Z = \int \mathcal{D}J e^{i \int d^2x J - \frac{3}{2} T R \log D}, \quad D_{AB}(x, y) = \delta^2(x - y) \left(-\frac{1}{4\lambda^2} \partial^2 \delta_{AB} + iJ \delta_{AB} \right) \quad (66)$$

This is exactly the partition function of an $O(M)$ non-linear sigma model in the limit $M \rightarrow \infty$ with $\bar{\lambda}^2 M$ held fixed equal to $6g_f^2$ where $M = 3(N^2 - 1) = 12|L|^2$. Indeed we have

$$Z = \int \mathcal{D}J \exp\left(\frac{i}{4\lambda^2} \int d^2x J - \frac{M}{2} \int d^2x \langle x | \log(-\partial^2 + iJ) | x \rangle\right). \quad (67)$$

All terms in the exponent are now of the same order M and thus the model can be solved using the method of steepest descents . Minimizing the exponent yields the equation

$$\langle x | \frac{1}{-\partial^2 + iJ} | x \rangle = \frac{1}{12g_f^2}. \quad (68)$$

Solutions $J(x)$ of this equation are obviously given by $J(x) = -im^2$ where m^2 are positive real constant numbers and thus this equation , which reads (in momentum space) $\int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + m^2} = \frac{1}{12g_f^2}$, admits the solutions

$$\frac{1}{12g_f^2} = \frac{1}{2\pi} \log \frac{\Lambda}{m} \quad (69)$$

where we have also regulated the integral with a momentum cutoff $\Lambda \gg m$. In order to get a regulator-independent coupling constant we will need to renormalize this theory and thus introduce explicitly a renormalization scale μ . This is achieved by the simple prescription

$$\frac{1}{12g_f^2} = \frac{1}{12g_r^2} + \frac{1}{2\pi} \log \frac{\Lambda}{\mu}. \quad (70)$$

From this result we can derive the beta function of the theory which we find to be given by

$$\beta(g_r) = \mu \frac{\partial g_r}{\partial \mu} = -\frac{3}{\pi} g_r^3. \quad (71)$$

This result , up to a numerical factor (which can always be understood as a normalization of the coupling constant) , is the same as the result (12) obtained in ordinary one-loop perturbation theory of the original Moyal-Weyl Plane . The crucial difference in this case is the fact that the above result is actually exact to all orders in $\bar{\lambda}^2 M = 6g_f^2$ and thus it is intrinsically nonperturbative [13]. The arbitrariness of the definition of the renormalized coupling constant is reflected in the fact that the solution of this theory depends on an arbitrary renormalization mass scale μ . Indeed it is not difficult to find that $m = m(g_r, \mu)$ is given by

$$m(g_r^2, \mu) = \mu e^{-\frac{\pi}{6g_r^2}}. \quad (72)$$

It is worth pointing out that this mass satisfies the Callan-Symanzik equation $[\mu \frac{\partial}{\partial \mu} + \beta(g_r) \frac{\partial}{\partial g_r}] m(g_r^2, \mu) = 0$ and hence everything is under control. Finally the non-vanishing of this $O(M)$ -mass means in particular that this $O(M)$ symmetry is never broken for all values of g_r^2 .

5 Conclusion

As we have discussed in this paper there are few problems with the path integral of field theory on the canonical noncommutative Moyal-Weyl spaces . The noncommutative plane is actually a zero-dimensional matrix model and not a continuum space. It acts however on an infinite dimensional Hilbert space and thus we are integrating in the path integral over infinite dimensional matrices which is a rather formal procedure . The second problem is the absence of rotational invariance due to the non-zero value of theta ; the noncommutativity parameter . A third problem is the appearance in the theory of a dimensionfull parameter , this same θ , which goes against the intuitive argument for this theory to be renormalizable .

The fuzzy sphere is a 0-dimensional matrix model with a gauge-invariant , Lorentz-invariant UV as well as IR cutoffs . In this approximation the noncommutative Moyal-Weyl planes can be simply viewed as large spheres (i.e with large radii R) which are represented by large but finite matrices (i.e with large representations L of $SU(2)$) . The relevant limit is a double scaling continuum planar limit where for example the ratio R/L is kept fixed equal to $\theta/2$ which is to be identified with the noncommutativity parameter . In this formulation it is obvious that the noncommutativity parameter θ^2 acquires its dimension of (length)² from the large radius of the underlying fuzzy approximation and hence renormalizability is not necessarily threatened .

In this article the above prescription is applied to 4-dimensional noncommutative $U(1)$ gauge theory with some remarkable results . For simplicity we have considered a minimal noncommutative space $\mathbf{R}^2 \times \mathbf{R}_\theta^2$. If we approximate this noncommutative spatial slice \mathbf{R}_θ^2 by a fuzzy sphere of matrix size L and radius R as explained above then the noncommutative degrees of freedom are converted into color degrees of freedom . Classically it is seen that the field theory on the fuzzy space $\mathbf{R}^2 \times \mathbf{S}_L^2$ reduces to the field theory on the Moyal-Weyl plane $\mathbf{R}^2 \times \mathbf{R}_\theta^2$ in the flattening continuum planar limits $R, L \rightarrow \infty$ where $R^2/(|L|^2)^q = \theta^2$. The effective noncommutativity parameter is however found to be given by $\theta_{eff}^2 \sim 2\theta^2 (\frac{L}{2})^{2q-1}$. In the quantum theory it turns out that this prescription is also equivalent to a dimensional reduction of the model where the noncommutative $U(1)$ gauge theory in 4 dimensions is shown to be equivalent in the large L limit to an ordinary $O(M)$ non-linear sigma model in 2 dimensions where $M = 12|L|^2$. More precisely the large L flattening planar limit is proven to be the same as t'Hooft limit of the $O(M)$ sigma model in which the coupling constant $\bar{\lambda} \rightarrow 0$ such that $M\bar{\lambda}^2$ is kept fixed equal to $6g_f^2$ where g_f is precisely the coupling constant of the original $U(1)$ theory . This result is only true for the class of scalings in which $q > \frac{3}{2}$ and where the corresponding Moyal-Weyl plane is strongly noncommuting . The model defined this way is also seen to be an ordinary renormalizable theory which can be solved exactly using the method of steepest descents to yield the beta function (71) . This beta function (71) agrees with the one-loop perturbative result (12) but as we have shown it is also an exact representation of the beta function of the theory to all orders in g_r^2 .

As we have said above the model can be solved exactly in the large L limit and one finds for a fixed renormalization scale μ and a fixed renormalized coupling g_r (or equivalently a fixed cut-off Λ and a fixed bare coupling g_f) a non-zero $O(M)$ -symmetric mass for the different M components of the sigma model field given by equation (72). This is clearly non-zero for all values of g_r^2 and hence the $O(M)$ symmetry is never broken in this solution .

Finally from the action (61) and from equation (26) we conclude that for the scalings $\frac{1}{2} < q < \frac{3}{2}$ we have a strongly noncommuting Moyal-Weyl plane where the action is dominated by the potential term , i.e the quantum description in this case is purely in terms of a matrix model . For $q < \frac{1}{2}$ the action is still dominated by the potential term but the Moyal-Weyl plane is weakly noncommuting . The values $q = \frac{1}{2}$ and $q = \frac{3}{2}$ are special. For $q = \frac{1}{2}$ the noncommutativity parameter is given by $\theta_{eff}^2 = 2\theta^2$ and the action is dominated by the potential term whereas for $q = \frac{3}{2}$ the Moyal-Weyl plane is strongly noncommuting but now both terms in the action (61) are important . The precise meaning of all this is still not clear .

Including non-trivial field configurations , such as those introduced in [14] , is still however an open question . Fermions and as a consequence chiral symmetry , in the sense of [15, 18] , are also not obvious how to formulate in this limit . Also since the fuzzy sphere parameter L is meant to be a cut-off we can ask the question how does the theory actually depends on L , in particular renormalizability of the $L = \infty$ is an open question. This is

obviously a much harder question and we are currently contemplating adapting the Polchinski approach to this problem . In 4–dimensions other choices for the fuzzy underlying manifolds are available such as fuzzy \mathbf{CP}^2 and fuzzy \mathbf{S}^4 but fuzzy $\mathbf{S}^2 \times \mathbf{S}^2$ seems much more practical as all the computation in the corresponding QFT's only involve the well known $SU(2)$ Clebsch-Gordan coefficients [16, 17].

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